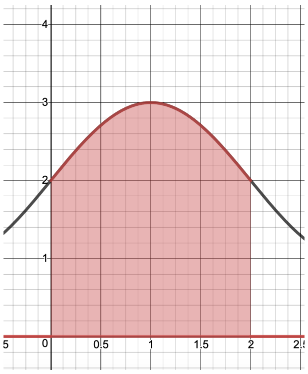
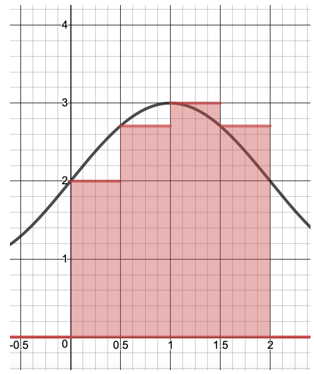
SECTION 8.8: NUMERICAL INTEGRATION

RECALL: Provided f is continuous, we can approximate $\int_a^b f(x)\,dx$ by using a:

- Left endpoint sum: LS_n

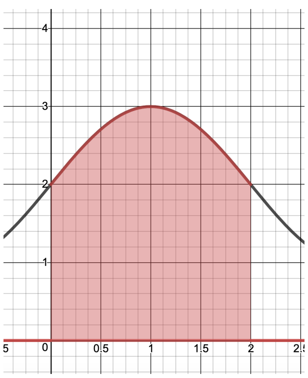


$$\int_0^2 f(x)\,dx$$

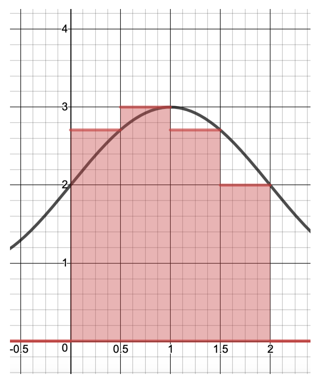


LS_4

- Right endpoint sum: RS_n

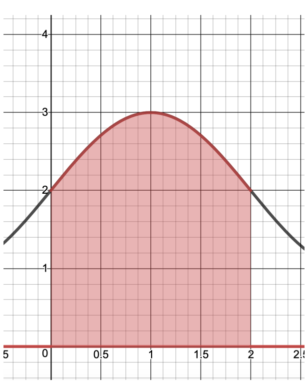


$$\int_0^2 f(x)\,dx$$

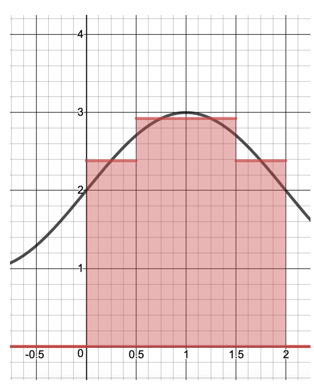


RS_4

- Midpoint sum: MS_n



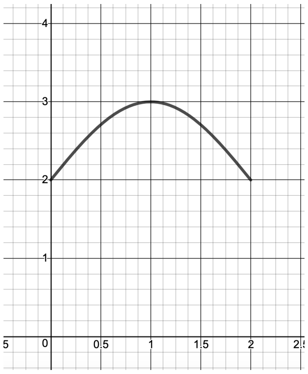
$$\int_0^2 f(x)\,dx$$



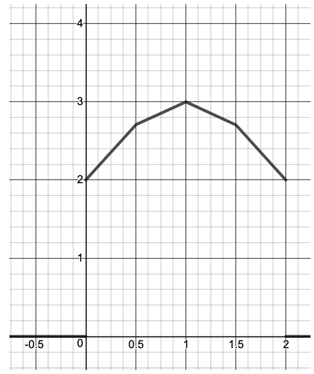
MS_4

THE TRAPEZOIDAL RULE

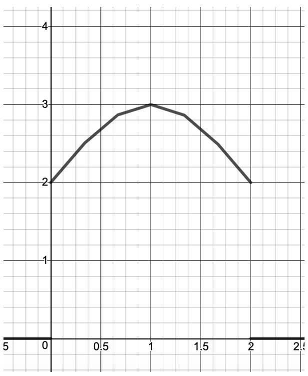
BIG IDEA: Instead of using **rectangles** formed by approximating $f(x)$ by a **constant** function $y = f(x_i^*)$ over the interval $[x_{i-1}, x_i]$, we can approximate $f(x)$ by the **secant line** over the interval $[x_{i-1}, x_i]$.



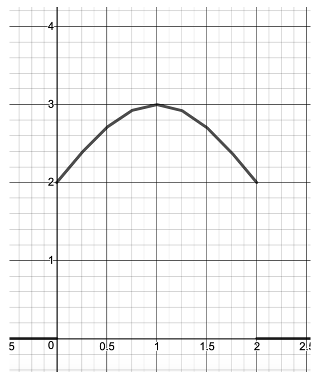
$y = f(x)$



Secant line segments, $n = 4$

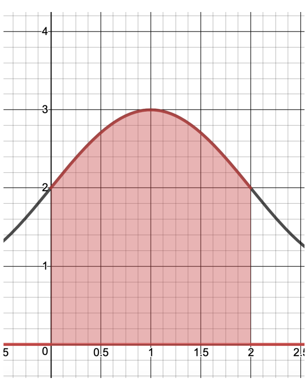


Secant line segments, $n = 6$

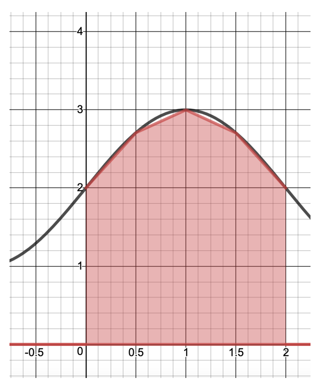


Secant line segments, $n = 8$

The secant line segments determine **trapezoids**. We can hence use the area of these trapezoids to approximate the area represented by the definite integral.



Area represented by $\displaystyle\int_0^2 f(x)\,dx$



Approximate area using 4 Trapezoids

DERIVING THE TRAPEZOIDAL RULE:

From geometry, we know the area of the ith trapezoid is: $A_i = \dfrac{1}{2}\left[f(x_{i-1}) + f(x_i)\right]\Delta x_i$.

NOTE: that this is none other than the average of the ith left and right sum rectangle!

For a regular partition, $\Delta x_i = \dfrac{b-a}{n}$ which means, after some reorganization: $A_i = \dfrac{b-a}{2n}\left[f(x_{i-1}) + f(x_i)\right]$

We define T_n to be the sum of the areas of the n trapezoids:

$$\begin{aligned}
T_n &= A_1 + A_2 + A_3 + \cdots + A_n \\
&= \frac{b-a}{2n}\left[f(x_0) + f(x_1)\right] + \frac{b-a}{2n}\left[f(x_1) + f(x_2)\right] + \frac{b-a}{2n}\left[f(x_2) + f(x_3)\right] + \ldots + \frac{b-a}{2n}\left[f(x_{n-1}) + f(x_n)\right] \\
&= \frac{b-a}{2n}\left[f(x_0) + f(x_1) + f(x_1) + f(x_2) + f(x_2) + f(x_3) + \ldots + f(x_{n-1}) + f(x_n)\right] \\
T_n &= \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \ldots + 2f(x_{n-1}) + f(x_n)\right]
\end{aligned}$$

EXAMPLE 1: Consider $f(x) = 4x - x^2$ over the interval $[0, 2]$. Recall we determined:

$LS_4 = 4.25$ $\qquad RS_4 = 6.25$ $\qquad \displaystyle\int_0^2 \left(4x - x^2\right)\, dx = 5.\overline{3}$

1. Approximate $\displaystyle\int_0^2 \left(4x - x^2\right)\, dx$ using the trapezoidal sum, T_4.

   Chopping $[0, 2]$ into 4 equal pieces gives $\Delta x = \frac{1}{2}$ so $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_2 = 1$, $x_3 = \frac{3}{2}$, and $x_4 = 2$.

   $$T_4 = \frac{2-0}{8}\left[f(0) + 2f\left(\frac{1}{2}\right) + 2f\left(1\right) + 2f\left(\frac{3}{2}\right) + f(2)\right] = \frac{1}{4}\left[0 + 2(1.75) + 2(3) + 2(3.75) + 4\right] = 5.25$$

   Hence, $\displaystyle\int_0^2 \left(4x - x^2\right)\, dx \approx T_4 = 5.25$ which is a reasonable approximation to the actual value of $5.\overline{3}$.

2. Verify T_4 is the average of LS_4 and RS_4: $\frac{1}{2}\left(LS_4 + RS_4\right) = \frac{1}{2}(4.25 + 6.25) = \frac{1}{2}(10.5) = 5.25 = T_4$ ✓

ERROR BOUND: If f'' is continuous on $[a, b]$, and $|f''(x)| \leq M$ for all x in $[a, b]$, then:

$$\left|\int_a^b f(x)\, dx - T_n\right| \leq \frac{M(b-a)^3}{12n^2}$$

EXAMPLE 2: Find n so that the trapezoidal approximation T_n is within 0.01 of $\displaystyle\int_0^\pi \sin(x)\, dx$.

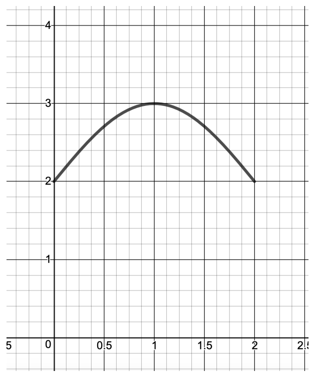
For $f(x) = \sin(x)$, $f''(x) = -\sin(x)$ so $|f''(x)| \leq 1$. Hence, we solve $\dfrac{M(b-a)^3}{12n^2} = \dfrac{\pi^3}{12n^2} \leq 0.01$.

We find $n \geq 16.07\ldots$ so $n = 17$ will do. We leave it to the reader to verify T_{17} is within 0.01 of $\displaystyle\int_0^\pi \sin(x)\, dx = 2$.

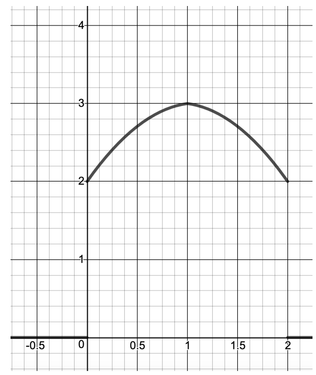
SIMPSON'S RULE

BIG IDEA: Instead of using **rectangles** (constant approximations) or **trapezoids** (linear approximations) what if we use **parabolic arcs** (quadratic approximations) to help us approximate definite integrals?

NOTE: Since parabolas $y = ax^2 + bx + c$ have three parameters, a, b, and c, we need three points to determine a parabola. Hence, we'll be using **two** subintervals for each parabolic arc: $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$.



$y = f(x)$



Four subintervals, two arcs.

AREA UNDER A PARABOLIC ARC: Using a 'healthy' amount of algebra along with the Fundamental Theorem of Calculus, it is possible to show that the area under the ith parabolic arc is given by:

$$A_i = \frac{b-a}{3n}\left[f(x_{i-1}) + 4f(x_i) + f(x_{i+1})\right]$$

We define S_n to be the sum of the areas under these $\frac{n}{2}$ parabolic arcs:

$$\begin{aligned}
S_n &= \frac{b-a}{3n}\left[f(x_0)+4f(x_1)+f(x_2)\right] + \frac{b-a}{3n}\left[f(x_2)+4f(x_3)+f(x_4)\right] + + \ldots + \frac{b-a}{3n}\left[f(x_{n-2})+4f(x_{n-1})+f(x_n)\right] \\
&= \frac{b-a}{3n}\left[f(x_0)+4f(x_1)+f(x_2)+f(x_2)+4f(x_3)+f(x_4)+\ldots+f(x_{n-2})+4f(x_{n-1})+f(x_n)\right] \\
S_n &= \frac{b-a}{3n}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\ldots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right]
\end{aligned}$$

NOTE: Simpson's Rule requires **two** subintervals for each parabolic arc, to find S_n, n must be **even**.

As with the Trapezoidal Rule, we have an error bound for Simpson's Rule:

ERROR BOUND: If $f^{(4)}(x)$ is continuous on $[a, b]$ and $|f^{(4)}(x)| \leq M$ for all x in $[a, b]$, then:

$$\left|\int_a^b f(x)\,dx - S_n\right| \leq \frac{M(b-a)^5}{180n^4}$$

NOTE: Since the error bound of Simpson's Rule is bounded by the fourth derivative of f, Simpson's Rule is **exact** for polynomials of degree three (or less) regardless of the number of parabolic arcs used! (Do you see why?) This means that we can find the definite integral for any degree 3 (or less) polynomial by evaluating S_2.

EXAMPLE 3: Consider $\int_0^{\pi} \sin(x)\,dx$.

- Find n so that the Simpson's Rule approximation S_n is within 0.01 of $\int_0^{\pi} \sin(x)\,dx$.

  Since $f^{(4)}(x) = \sin(x)$, $\left|f^{(4)}(x)\right| \le 1$ for all x in $[0, \pi]$.

  Solving $\dfrac{M(b-a)^5}{180n^4} = \dfrac{\pi^5}{180n^4} \le 0.01$ gives $n \ge 3.6\ldots$ so we choose $n = 4$.

  NOTE: When using trapezoids, we needed $n = 17$. Simpson's Rule saves us quite a bit of computation!

- Approximate $\int_0^{\pi} \sin(x)\,dx$ using Simpson's Rule using your answer above.

  $$S_4 = \frac{\pi - 0}{12}\left[f(0) + 4f\left(\frac{\pi}{4}\right) + 2f\left(\frac{\pi}{2}\right) + 4f\left(\frac{3\pi}{4}\right) + f(\pi)\right] = \frac{\pi}{12}\left[0 + 2\sqrt{2} + 2 + 2\sqrt{2} + 0\right] = \frac{\pi(1+2\sqrt{2})}{6}$$

  We find $\dfrac{\pi(1+2\sqrt{2})}{6} \approx 2.0045\ldots$ which is, indeed, within 0.01 of $\int_0^{\pi} \sin(x)\,dx = 2$.

EXAMPLE 4: The annual change in gross domestic product $G'(t)$ of a country (in million dollars per year) is:

year	$G'(t)$
2010	-4.6
2011	-1.7
2012	0.2
2013	0.8
2014	0.5
2015	1
2016	0.3

Use S_6 to approximate $\int_{2010}^{2016} G'(t)\,dt$.

$$\begin{aligned}
S_6 &= \frac{2016-2010}{3(6)}\left[G'(2010) + 4G'(2011) + 2G'(2012) + 4G'(2013) + 2G'(2014) + 4G'(2015) + G'(2016)\right] \\
&= \frac{1}{3}\left[(-4.6) + 4(-1.7) + 2(0.2) + 4(0.8) + 2(0.5) + 4(1) + (0.3)\right] \\
S_6 &= -0.8\overline{3}
\end{aligned}$$

This means the GDP of this country is (approximately) $0.8\overline{3}$ millions of dollars **less** in 2016 than it was in 2010.

HOMEWORK: Section 8.8: 19, 23, 35, 37, 39